An application of elementary functions to a resource allocation problem*

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In 1998, the West Report on tertiary education considered proposals for changing the proportion of funds given to universities on the basis of two criteria: research and teaching. An article by David Phillips, a former Head of the Higher Education Division of the Department of Employment, Education, Training and Youth Affairs, on the consequences of implementing these options, appeared in *The Australian* newspaper (see West, 1998 & Phillips, 1998).

Phillips considered the implications of increasing the total amount allocated to universities for research (called the "research quantum"). Assuming that the totality of funding to the universities for combined teaching and research purposes remained constant, this would have meant that the total allocation to the universities for teaching would have had to decrease. Phillips calculated the effects upon the total allocations to 36 universities for every 1% increase in the research quantum, pointing out that the effects could be significant. For example, for every 1% increase in the research quantum, he calculated that Melbourne University would gain $3.2 million and the University of Western Sydney would lose $1.6 million.

Mathematically, the problem considered by Phillips is one of resource allocation and analysing the changes and their implications when the resource allocation procedure is changed. This paper considers the original problem from a general and mathematical viewpoint, potentially applicable to problems other than the original one considered by Phillips. There is a changing environment in the allocation of resources in public policy, with more emphasis on allocating funds, status or recognition on the basis of specific criteria and performance. Even where the analysis in this paper is not directly applicable to all such problems, the modes of thought used here may illustrate the potential usefulness of mathematical thinking for general issues of public policy and resource allocation.

* This paper is a shortened version of a talk presented at the Mathematics Teachers' Day held at the University of Wollongong on 27 June 27 2006.

Statement of the problem

An allocating agency has a fixed amount of "money" or "recognition" which it allocates among recipients according to given criteria. For each criterion, a definite amount is allocated amongst the recipients on the basis of how well each recipient meets that criterion. The total amount set aside for allocation according to a given criterion reflects the importance the agency places upon that criterion in relation to the other criteria. The recipients may have strengths regarding some criteria and weaknesses regarding others, and these areas of strength and weakness may vary from one recipient to another. One day, the funding agency decides to alter the relative importance it places upon the criteria: some criteria are to be increased in importance, and others are to be decreased. However, the criteria themselves do not change. So, in relative terms, more is to be allocated on the basis of some of the criteria and less on the basis of some of the other criteria. Also, the allocating agency considers varying the total amount it allocates to the recipients. The following questions then arise.

1. How does the total allocation for each recipient change?
2. How does the proportional allocation for each recipient change?
3. Does the perceived status of a recipient change under the new allocation?
4. Would the changes have unforeseen or undesirable consequences and, if so, is there a different or fairer method of allocation which could be more suited to achieving desired outcomes?
5. Can we determine how much variation in outcomes can be achieved by varying the parameters of the process?

Of these questions, (1), (2) and (3) are discussed here in some detail, and some further comments are made that are also relevant to (4) and (5).

Mathematical formulation of the problem

In the analysis, we consider the allocation of funds to a number of recipients subject to two criteria, denoted by X and Y. Assume under criterion X that the total current allocation is A, and that under criterion Y the total current allocation is B. Then, if T is the total current allocation under both criteria,

$$T = A + B \tag{1}$$

Put

$$\rho = \frac{A}{B} \tag{2}$$

Then ρ measures the relative importance of criteria X and Y in the mind of the allocator. In the current allocation, assume that recipient j receives a_j under criterion X and b_j under criterion Y. It is convenient to assume that $a_j > 0$ and $b_j > 0$ for all j. Then

$$A = \sum_{j=1}^{n} a_j \quad \text{and} \quad B = \sum_{j=1}^{n} b_j \tag{3}$$

Put $$\rho_j = \frac{a_j}{b_j} \quad \text{for} \quad j = 1,\ 2,\ \ldots\ n \tag{4}$$

Then, ρ_j measures the extent to which recipient j meets criterion X compared with criterion Y, under the current allocation.

Now it could happen that

$$\rho_1 = \rho_2 = \ldots = \rho_n$$

This case is "trivial" in that there are no effective differences between the recipients on the basis of the criteria X and Y — there is, in effect, only one recipient. This case is not likely to arise in practice. The interesting case is when there are $j,k \in \{1, 2, \ldots, n\}$ such that

$$\rho_j \neq \rho_k$$

In this case, it can be deduced from (2), (3) and (4) that there are $j,k \in \{1, 2, \ldots, n\}$ such that

$$\rho_j < \rho < \rho_k \tag{5}$$

Now, due to changing circumstances, and wishing to encourage recipients to value one of the two criteria more than the other one, the allocator decides to change the relative importance of the two criteria. Furthermore, the allocator considers changing the total amount allocated under both criteria, and it is proposed to allocate a total amount T' instead of T. Thus, using (1), there is $\eta > 0$ such that

$$T' = \eta T = \eta A + \eta B$$

If $\eta > 1$, the total funding is increased, while if $\eta < 1$, the total funding is decreased. Also, in the new allocation there is an amount A' in place of A allocated under criterion X, and an amount B' in place of B under criterion Y. Thus,

$$T' = A' + B' = \eta A + \eta B \tag{6}$$

In the new allocations we assume that there is an actual change in the relative balance between the criteria X and Y, which means that

$$\frac{A'}{B'} \neq \frac{A}{B} \tag{7}$$

Also, there are $\theta,\phi > 0$ such

$$A' = \theta A \text{ and } B' = \phi B \tag{8}$$

We have from (7) and (8) that $\theta \neq \phi$, and from (6) and (8) that

$$(\theta - \eta)A = (\eta - \phi)B$$

Thus, either $\theta > \eta$ or $\phi > \eta$. The problem is symmetric in the criteria X and Y, so we may as well assume that $\theta > \eta$. Note that $\theta > \eta$ means that criterion X is made more important than criterion Y in relation to the new amount of total funds to be allocated under both criteria.

We have from (6) and (8) that

$$\phi = \frac{B'}{B} = \frac{\eta A + \eta B - \theta A}{B} = \eta\frac{A}{B} + \eta - \theta\frac{A}{B} = \eta\rho + \eta - \theta\rho \qquad (9)$$

Let a_j' = the new amount under criterion X for recipient j, and
b_j' = the new amount under criterion Y for recipient j.

The changed balance between X and Y is given by θ,ϕ as in (8), so we have also

$$a_j' = \theta a_j \text{ and } b_j' = \phi b_j \text{ for all } j \in \{1, 2, \ldots, n\} \qquad (10)$$

We have from (9) and (10) that

$$b_j' = (\eta\rho + \eta - \theta\rho)\, b_j, \text{ for all } j \in \{1, 2, \ldots, n\} \qquad (11)$$

We now consider how the allocations to recipients change. Using (4), (10) and (11) we see that the *absolute* change for recipient j is

$$\begin{aligned} a_j' + b_j' - a_j - b_j &= \theta a_j + (\eta\rho + \eta - \theta\rho) b_j - a_j - b_j \\ &= (\theta - 1) a_j + (\eta\rho + \eta - \theta\rho - 1) b_j \\ &= b_j\left((\theta - 1)\rho_j + \eta\rho + \eta - \theta\rho - 1\right) \qquad (12) \\ &= b_j\left((\theta - 1)(\rho_j - \rho) + (\eta - 1)(\rho + 1)\right) \qquad (13) \end{aligned}$$

Note that the absolute change for a recipient may be positive or negative — that is, a recipient may receive an increase or a decrease under the changed allocation procedure. However, in the case when $\eta = 1$, that is when $T' = T$ and there is no change in the total funding, (13) takes the simpler form

$$a_j' + b_j' - a_j - b_j = b_j(\theta - 1)(\rho_j - \rho)$$

Thus, when there is no change in the total funding and $\theta > 1$, we see that recipient j will receive an increase in funding when $\rho_j < \rho$, and a decrease in funding when $\rho_j > \rho$. This rather neat result is not surprising, but neither does it appear as completely obvious. Note that in this case, by (5), at least one recipient will receive an increase and at least one will receive a decrease.

What about the *proportional* change in the funding for recipients? Using (4) and (12) we see that for recipient j this is

$$\begin{aligned}\frac{a_j'+b_j'-a_j-b_j}{a_j+b_j} &= \left(\frac{b_j}{a_j+b_j}\right)\left((\theta-1)\rho_j+\eta\rho+\eta-\theta\rho-1\right)\\ &= \frac{(\theta-1)\rho_j+\eta\rho+\eta-\theta\rho-1}{\rho_j+1}\\ &= \frac{(\theta-1)(\rho_j+1)-\theta+\eta\rho+\eta-\theta\rho}{\rho_j+1}\\ &= (\theta-1)-(\theta-\eta)\left(\frac{\rho+1}{\rho_j+1}\right)\end{aligned} \tag{14}$$

This equation shows that the proportional change in allocation depends upon the ratio $\rho_j = a_j/b_j$, but not upon a_j or b_j directly. The interest in (14) is on how the value of ρ_j affects the proportional change in allocation for recipient j, treating the other parameters as fixed. Define a function P by

$$P(x) = (\theta-1)-(\theta-\eta)\left(\frac{\rho+1}{x+1}\right) \quad \text{for} \quad x>0$$

As we are taking $\theta > \eta$, we see that P is increasing in x. Since it is clear from (14) that $P(\rho_j)$ is the proportional change in the allocation for recipient j, we see that the recipients with higher values of ρ_j benefit more in proportional terms (see Figure 1, which shows the graph of P in the case $\eta = 1$).

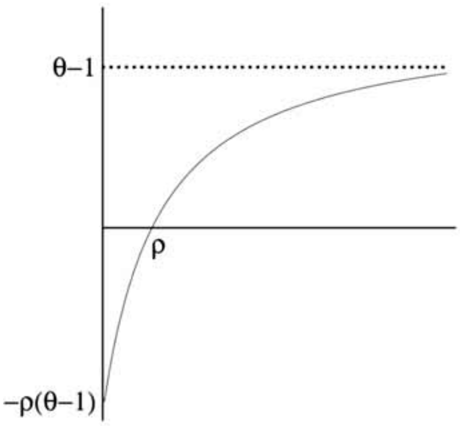

Figure 1. The graph of the function P in the case $\eta = 1$, where P is given by

$$P(x) = (\theta-1)-\frac{(\theta-1)(\rho+1)}{x+1}$$

Now, the derivative of P is given by

$$P'(x) = \frac{(\theta - \eta)(\rho + 1)}{(x+1)^2}$$

and we see that $P'(x)$ is decreasing in x. Thus, $P(x)$ changes more rapidly for small values of x, as separately indicated in Figure 1 for the case $\eta = 1$. So, for recipients who have a low value of ρ_j, any slight differences between the values of ρ_j have greater effects on the proportional changes under the new allocations; small differences in ρ_j will have larger proportional effects on the recipients with lower values of ρ_j in comparison to those with higher values. However, the extent of these effects depends upon the actual values of the parameters in the problem.

General comments and conclusion

The nature of the new allocation method considered here is one of straight "reward or punishment" — that is, those recipients who are stronger in relation to the newly preferred criterion are rewarded, while the others are punished. This method of proceeding does not encourage diversity, but positively discourages it. So, if the encouragement of diversity were an aim, there would need to be modification of the proposed reallocation method. Furthermore, in proportional terms, the method does not produce outcomes for those who perform better or worse at the preferred criterion in strict proportion to the extent to which they are better or worse at that criterion — that is, the proportional change for recipient j is not in proportion to $\rho_j - \rho$ but instead is determined by the value of

$$\frac{\rho + 1}{\rho_j + 1}$$

as we see in (14). This raises the question of the fairness of the allocation method and indicates the need for further analysis to understand better the effects of the new allocation method, and possible alternatives to it. Also, the recipients with lower values of ρ_j are more vulnerable to slight variations in the value of ρ_j. So, the analysis has revealed possibly negative features that were not evident in the original description of the proposal.

The new allocations may produce a situation where it is considered that too many recipients receive a decrease in allocation. The analysis can then be used to estimate by how much the total funds would need to be increased to give a specified number of recipients an increased allocation. In particular, if we put

$$\sigma = \min\{\rho_1, \rho_2, \ldots, \rho_n\}$$

then it follows from (13) that every recipient will receive an increase precisely when

$$\eta > 1 + (\theta - 1)\left(\frac{\rho - \sigma}{\rho + 1}\right)$$

The analysis also reveals precisely how the outcomes of the new procedure depend upon the given parameters.

More generally, in any mathematical analysis of actual procedures the underlying assumptions in the procedures may appear to be reasonable and fair, but the analysis may reveal implications which were not apparent in the original assumptions. Such implications may even be inconsistent with the intentions behind the changes, and may suggest that alternative procedures should be considered. The question then arises as to the extent to which the procedures or parameters can be changed or "manipulated" to avoid unacceptable outcomes. Conceptually, we can think of this as interchanging the roles of the input and output parameters.

We must be aware that a procedure is not grasped merely by understanding the immediate techniques of how to carry it out. We grasp a procedure better when we can comprehend its effects as a whole, and understand how outcomes vary with variation in the parameters. Then, we may grasp the procedure better still when we can assess its range of possible outcomes against other alternative procedures. Such an approach requires a conscious use of mathematics as more than a mere tool of calculation, but rather as a precise means of critical reflection and of exploring possibilities. It also requires us to "distance" ourselves from the procedure, and to consider it dispassionately as to its fairness and appropriateness. When pursued at a sufficiently high or intense level, involving the whole person, this type of analysis shatters the mental barrier, common in Australia, which limits education to the acquisition of information. Once the whole person becomes involved, the analysis can take on an ethical and moral dimension whose justification lies beyond its immediate aims, an effect accentuated in mathematics because of the objectivity adhering to it, and the "distancing" of the argument from the personal wishes of the analyst.

In its concern with the application of elementary mathematics to social policy, the spirit of this paper is similar to that of Nillsen (2007), and further information on applying mathematics to public policy is on the author's website (www.uow.edu.au/~nillsen). This website includes a copy of the article by Phillips(1998). I am indebted to David Phillips for giving me permission to make his article available in this way.

References

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[Recommendation 37 of the report was: "that the Government review the size and basis for allocating the research quantum." Page 164 of the report considers 4.9%, 6% and 10% as possible proportions of the total budget for universities that could be based upon research.]